ON MAXIMAL REGULARITY AND SEMIVARIATION OF α-TIMES RESOLVENT FAMILIES

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ABSTRACT. Let $1 < \alpha < 2$ and A be the generator of an α-times resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X. It is shown that the fractional Cauchy problem $\mathbf{D}_t^\alpha u(t) = Au(t) + f(t)$, $t \in [0,r]$; $u(0), u'(0) \in D(A)$ has maximal regularity on $C([0,r];X)$ if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0,r]$.

1. INTRODUCTION

Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

$$\begin{aligned} &u'(t) = Au(t) + f(t), \quad t \in [0,r] \\ &u(0) = x \in D(A) \end{aligned} \tag{1.1}$$

where A is the generator of a C_0-semigroup. One says that (1.1) has maximal regularity on $C([0,r];X)$ if for every $f \in C([0,r];X)$ there exists a unique $u \in C^1([0,r];X)$ satisfying (1.1). From the closed graph theorem it follows easily that if there is maximal regularity on $C([0,r];X)$, then there exists a constant $C > 0$ such that

$$\|u'\|_{C([0,r];X)} + \|Au\|_{C([0,r];X)} \leq \|f\|_{C([0,r];X)}.$$

Travis [5] proved that the maximal regularity is equivalent to the C_0-semigroup generated by A being of bounded semivariation on $[0,r]$.

Chyan, Shaw and Piskarev [2] gave similar results for second order Cauchy problems. More precisely, they showed that the second order Cauchy problem

$$\begin{aligned} &u''(t) = Au(t) + f(t), \quad t \in [0,r] \\ &u(0) = x,\, u'(0) = y, \quad x, y \in D(A) \end{aligned} \tag{1.2}$$

has maximal regularity on $[0,r]$ if and only if the cosine opeator function generated by A is of bounded semivariation on $[0,r]$.

In this paper we will consider the maximal regularity for fractional Cauchy problem

$$\begin{aligned} &\mathbf{D}_t^\alpha u(t) = Au(t) + f(t), \quad t \in [0,r] \\ &u(0) = x,\, u'(0) = y, \quad x, y \in D(A) \end{aligned} \tag{1.3}$$

where $\alpha \in (1,2)$, A is the generator of an α-times resolvent family (see Definition 2.2 below) and $\mathbf{D}_t^\alpha u$ is understood in the Caputo sense. We show that (1.3) has maximal regularity on $C([0,r];X)$ if and only if the corresponding α-times resolvent family is of bounded semivariation on $[0,r]$.

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2. Preliminaries

Let $1 < \alpha < 2$, $g_0(t) := \delta(t)$ and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)} (\beta > 0)$ for $t > 0$. Recall the Caputo fractional derivative of order $\alpha > 0$

$$\mathbf{D}_t^\alpha f(t) := \int_0^t g_{2-\alpha}(t-s) \frac{d^2}{ds^2} f(s) ds, \quad t \in [0, r]$$

for $f \in C^2([0,r]; X)$. The condition that $f \in C^2([0,r]; X)$ can be relaxed to $f \in C^1([0,r]; X)$ and $g_{2-\alpha} * (f - f(0) - f'(0)g_2) \in C^2([0,r]; X)$, for details and further properties see [1] and references therein. And in the above we denote by

$$(g_\beta * f)(t) = \int_0^t g_\beta(t-s) f(s) ds$$

the convolution of g_β with f. Note that $g_\alpha * g_\beta = g_{\alpha+\beta}$.

Consider a closed linear operator A densely defined in a Banach space X and the fractional evolution equation (1.3).

Definition 2.1. A function $u \in C([0,r]; X)$ is called a *strong solution* of (1.3) if

$$u \in C([0,r]; D(A)) \cap C^1([0,r]; X), \quad g_{2-\alpha} * (u(t) - x - ty) \in C^2([0,r]; X)$$

and (1.3) holds on $[0,r]$. $u \in C([0,r]; X)$ is called a *mild solution* of (1.3) if $g_\alpha * u \in D(A)$ and

$$u(t) - x - ty = A(g_\alpha * u)(t) + (g_\alpha * f)(t)$$

for $t \in [0,r]$.

Definition 2.2. Assume that A is a closed, densely defined linear operator on X. A family $\{S_\alpha(t)\}_{t\geq 0} \subset B(X)$ is called an *α-times resolvent family* generated by A if the following conditions are satisfied:

(a) $S_\alpha(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;
(b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A), t \geq 0$;
(c) For all $x \in D(A)$ and $t \geq 0$, $S_\alpha(t)x = x + (g_\alpha * S_\alpha)(t)Ax$.

Remark 2.3. Since A is closed and densely defined, it is easy to show that for all $x \in X$, $(g_\alpha * S_\alpha)(t)x \in D(A)$ and $A(g_\alpha * S_\alpha)(t)x = S_\alpha x - x$.

The alpha-times resolvent families are closely related to the solutions of (1.3). It was shown in [1] that if A generates an α-times resolvent family $S_\alpha(\cdot)$, then (1.3) has a unique strong solution given by $S_\alpha(t)x + \int_0^t S_\alpha(s)yds$.

Next we recall the definition of functions of bounded semivariation (see e.g. [3]). Given a closed interval $[a,b]$ of the real line, a subdivision of $[a,b]$ is a finite sequence $d : a = d_0 < d_1 < \cdots < d_n = b$. Let $D[a,b]$ denote the set of all subdivisions of $[a,b]$.

Definition 2.4. For $G : [a,b] \to B(X)$ and $d \in D[a,b]$, define

$$SV_d[G] = \sup\{\| \sum_{n=1}^{n} [G(d_i) - G(d_{i-1})]x_i\| : x_i \in X, \|x_i\| \leq 1\}$$

and $SV[G] = \sup\{SV_d[G] : d \in D[a,b]\}$. We say G is of bounded sevivariation if $SV[G] < \infty$.

3. Main results

We begin with some properties on α-times resolvent families which will be needed in the sequel.

Proposition 3.1. *Let $1<\alpha<2$ and $\{S_\alpha(t)\}_{t\geq 0}$ be the α-times resolvent family with generator A. Define*

$$P_\alpha(t)x=(g_{\alpha-1}*S_\alpha)(t)x=\int_0^t g_{\alpha-1}(t-s)S_\alpha(s)xds, \quad x\in X,$$

then the following statements are true.

(a) For every $x\in X$, $\int_0^t P_\alpha(s)xds\in D(A)$ and

$$A\int_0^t P_\alpha(s)xds=S_\alpha(t)x-x;$$

(b) For every $x\in X$, $0\leq a,b\leq t$, $\int_a^b sP_\alpha(t-s)xdx\in D(A)$ and

$$A\int_a^b sP_\alpha(t-s)xds=aS_\alpha(t-a)x-bS_\alpha(t-b)x+\int_a^b S_\alpha(t-s)xds;$$

(c) For every $x\in X$, $\int_0^t g_\alpha(t-s)sP_\alpha(s)xds\in D(A)$ and

$$A\Big(\int_0^t g_\alpha(t-s)sP_\alpha(s)xds\Big)=-\alpha(g_\alpha*S_\alpha)(t)x+tP_\alpha(t)x;$$

(d) If $f\in C([0,r];X)$, then $g_\alpha*S_\alpha*f\in D(A)$ and

$$A(g_\alpha*S_\alpha*f)=(S_\alpha-1)*f. \tag{3.1}$$

Proof. (a) follows from the fact that $\int_0^t P_\alpha(s)xds=(g_1*g_{\alpha-1}*S_\alpha)(t)x=(g_\alpha*S_\alpha)(t)x\in D(A)$ and $A(g_\alpha*S_\alpha)(t)x=S_\alpha(t)x-x$ by Remark 2.3.

(b) By integration by parts we have

$$\begin{aligned}\int_a^b sP_\alpha(t-s)xds &= \int_a^b sd_s[\int_0^s P_\alpha(t-\tau)xd\tau]\\ &= \int_a^b sd_s[(g_\alpha*S_\alpha)(t-s)x]\\ &= -s(g_\alpha*S_\alpha)(t-s)x\Big|_a^b+\int_a^b(g_\alpha*S_\alpha)(t-s)xds\\ &= a(g_\alpha*S_\alpha)(t-a)x-b(g_\alpha*S_\alpha)(t-b)x+\int_a^b(g_\alpha*S_\alpha)(t-s)xds,\end{aligned}$$

since $(g_\alpha*S_\alpha)(t)xds\in D(A)$ by Remark 2.3, operating A on both sides of the above identity gives (b).

(c) follows from the fact that

$$\begin{aligned} & \int_0^t g_\alpha(t-s)sP_\alpha(s)xds \\ = & \int_0^t g_\alpha(t-s)(s-t)P_\alpha(s)xds + t\int_0^t g_\alpha(t-s)P_\alpha(s)xds \\ = & -\alpha\int_0^t g_{\alpha+1}(t-s)P_\alpha(s)xds + t(g_\alpha * P_\alpha)(t)x \\ = & -\alpha(g_{\alpha+1} * P_\alpha)(t)x + t(g_\alpha * P_\alpha)(t)x \\ = & -\alpha(g_{\alpha+1} * g_{\alpha-1} * S_\alpha)(t)x + t(g_\alpha * g_{\alpha-1} * S_\alpha)(t)x \\ = & -\alpha(g_\alpha * g_\alpha * S_\alpha)(t)x + t(g_{\alpha-1} * g_\alpha * S_\alpha)(t)x \end{aligned}$$

belongs to $D(A)$ and

$$\begin{aligned} A(\int_0^t g_\alpha(t-s)sP_\alpha(s)xds) &= -\alpha(g_\alpha * A(g_\alpha * S_\alpha))(t)x + t(g_{\alpha-1} * A(g_\alpha * S_\alpha))(t)x \\ &= -\alpha(g_\alpha * (S_\alpha - 1))(t)x + t(g_{\alpha-1} * (S_\alpha - 1))(t)x \\ &= -\alpha(g_\alpha * S_\alpha)(t)x + \alpha g_{\alpha+1}(t)x + t(g_{\alpha-1} * S_\alpha)(t) - tg_\alpha(t)x \\ &= -\alpha(g_\alpha * S_\alpha)(t)x + tP_\alpha(t)x. \end{aligned}$$

(d) (3.1) is true for step functions, and then for continuous functions by the closedness of A. □

The following two lemmas can be proved similarly as that in [2, 5].

Lemma 3.2. *If $f \in C([0,r];X)$ and the α-times resolvent family $S_\alpha(t)$ is of bounded semivariation on $[0,r]$, then $(P_\alpha * f)(t) \in D(A)$ and*

$$A(P_\alpha * f)(t) = -\int_0^t d_s[S_\alpha(t-s)]f(s).$$

Lemma 3.3. *If $f \in C([0,r];X)$ and the α-times resolvent family $S_\alpha(t)$ is of bounded semivariation on $[0,r]$, then $\int_0^t d_s[S_\alpha(t-s)]f(s)$ is continuous in t on $[0,r]$.*

We next turn to the solution of

$$\begin{aligned} &\mathbf{D}_t^\alpha u(t) = Au(t) + f(t), \quad t \in [0,r], \\ &u(0) = 0,\ u'(0) = 0, \end{aligned} \tag{3.2}$$

where A is the generator of an α-times resolvent family. If $v(t)$ is a mild solution of (3.2), then by Definition 2.1 $(g_\alpha * v)(t) \in D(A)$ and $v(t) = A(g_\alpha * v)(t) + (g_\alpha * f)(t)$. It then follows from the properties of α-times resolvent family that

$$1 * v = (S_\alpha - A(g_\alpha * S_\alpha)) * v = S_\alpha * v - S_\alpha * A(g_\alpha * v) = S_\alpha * (v - A(g_\alpha * v)) = S_\alpha * g_\alpha * f,$$

which implies that $g_\alpha * S_\alpha * f$ is differentiable and

$$v(t) = \frac{d}{dt}(g_\alpha * S_\alpha * f)(t) = (g_{\alpha-1} * S_\alpha * f)(t) = (P_\alpha * f)(t).$$

Therefore, the mild solution of (1.3) is given by

$$u(t) = S_\alpha(t)x + \int_0^t S_\alpha(s)yds + (P_\alpha * f)(t). \tag{3.3}$$

Proposition 3.4. *Let A be the generator of an α-times resolvent family $S_\alpha(\cdot)$, and let $f \in C([0,r];X)$ and $x, y \in D(A)$. Then the following statements are equivalent:*
(a) (1.3) has a strong solution;
(b) $(S_\alpha * f)(\cdot) \in C^1([0,r];X)$;
(c) $(P_\alpha * f)(t) \in D(A)$ for $0 \le t \le r$ and $A(P_\alpha * f)(t)$ is continuous in t on $[0,r]$.

Proof. (a) If $u(t)$ is a strong solution of (1.3), then u is given by (3.3) since every strong solution is a mild solution. Therefore, by the definition of strong solutions, $g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f \in C^2([0,r];X)$; it then follows that $S_\alpha * f \in C^1([0,r];X)$, this is (b).

$(b) \Rightarrow (c)$. Suppose that $S_\alpha * f \in C^1([0,r];X)$. Since $g_1 * P_\alpha * f = g_\alpha * S_\alpha * f$, by Proposition 3.1(d), $g_1 * P_\alpha * f \in D(A)$ and

$$A(g_1 * P_\alpha * f) = A(g_\alpha * S_\alpha * f) = (S_\alpha - 1) * f. \tag{3.4}$$

Since A is closed and $S_\alpha * f \in C^1([0,r];X)$, we have $P_\alpha * f \in D(A)$ and $A(P_\alpha * f) = (S_\alpha * f)' - f$ is continuous.

$(c) \Rightarrow (a)$. By (3.4), $g_1 * A(P_\alpha * f) = A(g_1 * P_\alpha * f) = (S_\alpha - 1) * f$, therefore $S_\alpha * f$ is differentiable and thus $g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f$ is in $C^2([0,r];X)$. It is easy to check that $u(t)$ defined by (3.3) is a strong solution of (1.3). □

Now we are in the position to give the main result of this paper. The proof is similar to that of Proposition 3.1 in [5] or Theorem 4.2 in [2], we write it out for completeness.

Theorem 3.5. *Suppose that A generates an α-times resolvent family $\{S_\alpha(t)\}_{t\ge 0}$. Then the function (3.3) is a strong solution of the Cauchy problem (1.3) for every pair $x, y \in D(A)$ and continuous function f if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0,r]$.*

Proof. The sufficiency follows from Lemmas 3.2 and 3.3.

Conversely, suppose that for $x, y \in D(A)$ and continuous function f, $u(t)$ given by (3.3) is a strong solution for (1.3). Define the bounded linear operator $L : C([0,r];X) \to X$ by $L(f) = (P_\alpha * f)(r)$. By Proposition 3.4 (c) $Lf \in D(A)$, it thus follows from the closedness of A that $AL : C([0,r];X) \to X$ is bounded.

Let $\{d_i\}_{i=0}^n$ be a subdivision of $[0,r]$ and $\epsilon > 0$ such that $\epsilon < \min_{1\le i\le n}\{|d_i - d_{i-1}|\}$. For $x_i \in X$ with $\|x_i\| \le 1$ $(i = 1, 2, \cdots, n+1)$, define $f_{d,\epsilon} \in C([0,r];X)$ by

$$f_{d,\epsilon}(\tau) = \begin{cases} x_i, & d_{i-1} \le \tau \le d_i - \epsilon \\ x_{i+1} + \frac{\tau - d_i}{\epsilon}(x_{i+1} - x_i), & d_i - \epsilon \le \tau \le d_i \end{cases},$$

then $\|f_{d,\epsilon}\|_{C([0,r];X)} \le 1$. By Proposition 3.1,

$$\begin{aligned} AL(f_{d,\epsilon}) &= A\int_0^r P_\alpha(r-s) f_{d,\epsilon}(s)ds \\ &= \sum_{i=1}^n \Big[A\int_{d_{i-1}}^{d_i-\epsilon} P_\alpha(r-s)x_i ds \\ &\quad + A\int_{d_i-\epsilon}^{d_i} P_\alpha(r-s)x_{i+1}ds + A\int_{d_i-\epsilon}^{d_i} \frac{s-d_i}{\epsilon} P_\alpha(r-s)(x_{i+1}-x_i)dx\Big] \end{aligned}$$

$$
\begin{aligned}
= \quad & \sum_{i=1}^{n}\Big\{[S_\alpha(r-d_{i-1})x_i - S_\alpha(r-d_i+\epsilon)x_i] \\
& +[S_\alpha(r-d_i+\epsilon)x_{i+1} - S_\alpha(r-d_i)x_{i+1}] \\
& -\frac{d}{\epsilon}[S_\alpha(r-d_i+\epsilon)(x_{i+1}-x_i) - S_\alpha(r-d_i)(x_{i+1}-x_i)] \\
& +\frac{1}{\epsilon}[(d_i-\epsilon)S_\alpha(r-d_i+\epsilon)(x_{i+1}-x_i) - d_iS_\alpha(r-d_i)(x_{i+1}-x_i)] \\
& +\frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)ds\Big\} \\
= \quad & \sum_{i=1}^{n}\Big\{[S_\alpha(r-d_{i-1})x_i - S_\alpha(r-d_i)x_{i+1}] \\
& +\frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)ds\Big\} \\
= \quad & \sum_{i=1}^{n}\Big\{[S_\alpha(r-d_{i-1}) - S_\alpha(r-d_i)]x_i - S_\alpha(r-d_i)(x_{i+1}-x_i) \\
& +\frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)ds\Big\},
\end{aligned}
$$

it then follows that

$$
\begin{aligned}
& \Big\|\sum_{i=1}^{n}[S_\alpha(r-d_{i-1}) - S_\alpha(r-d_i)]x_i\Big\| \\
\leq \quad & \|AL(f_{d,\epsilon})\| + \sum_{i=1}^{n}\Big\|S_\alpha(r-d_i)(x_{i+1}-x_i) - \frac{1}{\epsilon}\int_{d_i-\epsilon}^{d_i} S_\alpha(r-s)(x_{i+1}-x_i)ds\Big\|.
\end{aligned}
$$

By letting $\epsilon \to 0$, we obtain that S_α is of bounded semivariation on $[0, r]$. □

Corollary 3.6. *Suppose that $\{S_\alpha(t)\}_{t\geq 0}$ is an α-times resolvent family with generator A and $S_\alpha(\cdot)$ is of bounded semivariation on $[0, r]$ for some $r > 0$. Then $R(P_\alpha(t)) \subset D(A)$ for $t \in [0, r]$ and $\|tAP_\alpha(t)\|$ is bounded on $[0, r]$.*

Proof. For $x \in X$, consider $f(t) = \alpha S_\alpha(t)x$. By Proposition 3.1(c), $tP_\alpha(t)x$ is a mild solution of (3.2). Moreover, it follows from Proposition 3.4 that $P_\alpha * f$ is a strong solution of (3.2). Since a strong solution must be a mild solution, we have $(P_\alpha * f)(t) = tP_\alpha(t)x$. Thus our claim follows from Proposition 3.4. □

Remark 3.7. Let $\alpha = 1$. If A generates a C_0-semigroup $T(\cdot)$, then the condition that $tAT(t)$ is bounded on $[0, r]$ implies that $T(\cdot)$ is analytic (see [4]). When $\alpha = 2$ and A generates a cosine function $C(\cdot)$, then the condition that $tAC(t)$ is bounded on $[0, r]$ implies that A is bounded ([2]). However, since there is no semigroup properties for α-times resolvent family, it is not clear that one can get the analyticity of $S_\alpha(\cdot)$ from the local boundedness of $tAP_\alpha(t)$.

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